THE LEBESGUE DECOMPOSITION OF MEASURES ON ORTHOMODULAR POSETS

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1. Introduction

THE study of measures on orthomodular posets has its origin in the quantum logic approach to quantum mechanics [**2**, **5**, **6**, **12**, **15**] and as a mathematical branch it became known as non-commutative measure theory (e.g. see [**17**]).

The purpose of this paper is to investigate the Lebesgue decomposition of positive measures into positive measures in this non-commutative setting. The geometrical aspect of this notion is emphasized.

The main theorems in §3 (3.4 & 3.5) present a condition for subcones of the cone of positive measures on an orthocomplete orthomodular poset L under which the requirement of a positive Lebesgue decomposition is equivalent to the poset L to be a Boolean lattice. These conditions are met by the cone obtained by restricting normal positive linear functionals on a JBW-algebra to the complete orthomodular lattice of idempotents in this algebra. As an application of the aforementioned result we obtain a measure-theoretic characterization of associative JBW-algebras amongst all JBW-algebras. This is the main result of §5. Paragraph 4 is concerned with certain permanence properties of the positive Lebesgue decomposition, i.e. the behaviour of this decomposition for selected sets of probability measures under the formation of direct products and direct sums. In particular it is shown that the collection of all probability measures of a finite constructible orthomodular poset has the positive Lebesgue decomposition property.

2. Prerequisites

Let us begin with the definitions and basic facts which pertain to this paper.

Let $(L, \leqslant, ')$ be an orthocomplemented poset, $^{\#}L > 1$, with 0 as the least and 1 as the greater element. A pair (p, q) of elements of L is said to be *orthogonal*, denoted by $p \perp q$, provided $p \leqslant q'$. A subset D of L is said to be *orthogonal* if each pair (p, q) of D with $p \neq q$ is orthogonal. Clearly, a subset C of L has at most one supremum, resp. infimum. We denote the supremum of C, resp. infimum of C, if such exists, by

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$\sup C = \bigvee C$, resp. $\inf C = \bigwedge C$. The following notations are also used: $p \vee q = \sup \{p, q\}$, $p \wedge q = \inf \{p, q\}$, $\bigvee_{i \in I} p_i = \sup \{p_i\colon i \in I\}$, $\bigwedge_{i \in I} p_i = \inf \{p_i\colon i \in I\}$.

By an *orthomodular poset* [4, 13, 17] we mean an orthocomplemented poset $(L, \leq, ')$ satisfying the following conditions:

(i) if $p \perp q$ then $p \vee q$ exists,

(ii) if $p \perp q$ and $p \vee q = 1$ then $p = q'$.

In an orthocomplemented poset and in presence of (i) condition (ii) is equivalent with

(ii′) if $p \leq q$ then $q = p \vee (q \wedge p')$,

noticing that the right-hand side exists.

An orthomodular poset is said to be *orthocomplete*, resp. *σ-orthocomplete*, if the supremum of every orthogonal subset, resp. countable orthogonal subset, exists. If an orthomodular poset is indeed a lattice then it is referred to as an *orthomodular lattice*. Notice that an orthocomplete, resp. σ-orthocomplete, orthomodular lattice is complete, resp. σ-complete [9, 21].

Let $(L, \leq, ')$ be an orthomodular poset. A pair (p, q) of elements of L is said to be *compatible*, denoted by pCq, provided there exist elements u, v and w with $u \perp v \perp w \perp u$ such that

$$p = u \vee v$$

and

$$q = u \vee w.$$

Notice that for elements p and q of L

$$pCq \Leftrightarrow qCp \Leftrightarrow pCq'$$

holds true. Also, pCq if and only if $p \wedge q$ and $p \wedge q'$ both exist and

$$p = (p \wedge q) \vee (p \wedge q').$$

If all pairs of elements of L are compatible then the orthomodular poset $(L, \leq, ')$ is said to be *Boolean*. One verifies that in this case $(L, \leq, ')$ is indeed a Boolean lattice.

Let $(L, \leq, ')$ be an orthomodular poset. An element μ of the product vector space $\mathbb{R}^L$ is said to be a *measure on L* if $p \perp q$ implies that

$$\mu(p \vee q) = \mu(p) + \mu(q).$$

A measure μ is said to be *positive* if $\mu(p) \geq 0$ for all $p \in L$. The collection of positive measures on L, denoted by $K(L)$, forms a cone in $\mathbb{R}^L$. By orthomodularity, a positive measure on L is an isotonic map on the poset

$(L, \leqslant)$. A measure μ is said to be *normalized* if

$$\mu(1) = 1.$$

Finally, a positive and normalized measure is called a *probability measure* on L. The collection of probability measures on L, denoted by $\Omega(L)$, is a convex and τ-compact subset of $\mathbb{R}^L$ where τ is the product topology on $\mathbb{R}^L$. Moreover, $K(L)$ coincides with the positive hull of $\Omega(L)$.

A measure μ is called *completely additive*, resp. *σ-additive*, if for every orthogonal, resp. countable orthogonal, subset D of L for which $\bigvee D$ exists

$$\mu\left(\bigvee D\right) = \lim \left(\mu\left(\bigvee C\right)\right)_{C \in D^f}$$

holds true, where $(D^f, \subseteq)$ denotes the collection of finite subsets of D directed by set-inclusion. With $\Omega_c(L)$, resp. $\Omega_\sigma(L)$, we denote the set of completely additive, resp. σ-additive, probability measures on L. Notice that both $\Omega_c(L)$ and $\Omega_\sigma(L)$ are faces of $\Omega(L)$. For details of these and other properties of measures on orthomodular posets the reader is referred to [**3**, **19**].

A positive measure μ on an orthomodular poset $(L, \leqslant, ')$ is said to be a *Jauch-Piron measure* [**18**] provided that every finite subset of the kernel of μ, denoted $\ker \mu$, has an upper bound in $\ker \mu$. The collection of normalized Jauch-Piron measures is denoted by $\Omega_{JP}(L)$. If $(L, \leqslant, ')$ is an orthomodular lattice and $\mu \in \Omega(L)$, then $\mu \in \Omega_{JP}(L)$ if and only if

$$\mu(p) = \mu(q) = 1$$

implies that

$$\mu(p \wedge q) = 1.$$

Accordingly, we define a positive measure μ to be a *c-Jauch-Piron*, resp. *σ-Jauch-Piron, measure* if every subset, resp. every countable subset, of $\ker \mu$ has an upper bound in $\ker \mu$. Then $\Omega_{c\text{-}JP}(L)$, resp. $\Omega_{\sigma\text{-}JP}(L)$, denotes the collection of all normalized c-Jauch-Piron, resp. σ-Jauch-Piron, measures on L.

Relations between the various notions of a Jauch-Piron measure and higher-order additivity of probability measures are given below. Notice that part (ii) is an extension of [**10**, Theorem 9].

THEOREM 2.1. *Let $(L, \leqslant, ')$ be an orthomodular poset.*

(i) *If $(L, \leqslant, ')$ is σ-orthocomplete then*

$$\Omega_\sigma(L) \cap \Omega_{JP}(L) \subseteq \Omega_{\sigma\text{-}JP}(L);$$

(ii) *if $(L, \leqslant, ')$ is orthocomplete then*

$$\Omega_c(L) \cap \Omega_{JP}(L) \subseteq \Omega_{c\text{-}JP}(L);$$

(iii) *if* $(L, \leqslant, ')$ *is orthocomplete or is a σ-complete orthomodular lattice (equivalently: σ-orthocomplete orthomodular lattice) then*

$$\Omega_\sigma(L) \cap \Omega_{c\text{-}JP}(L) \subseteq \Omega_c(L)$$

Proof. (i) Let μ be an element of $\Omega_\sigma(L) \cap \Omega_{JP}(L)$ and let E be a countable non-empty subset of $\ker \mu$. Let $(p_i)_{i \in \mathbb{N}}$ be an enumeration of E. We define the isotone sequence $(q_i)_{i \in \mathbb{N}}$ as follows:

$$q_1 := p_1 \quad \text{and} \quad q_i \in \ker \mu \quad \text{with} \quad q_i \geqslant q_1, q_2, \ldots, q_{i-1}, p_i \qquad \text{for} \quad i = 2, 3, \ldots .$$

Moreover, we define

$$r_1 := q_1, \quad r_i := q_i \wedge q'_{i-1} \quad \text{for} \quad i = 2, 3, \ldots .$$

Now $r_i \perp r_j$ provided that $i \neq j$ and

$$q_j = \bigvee_{i=1}^{j} r_i$$

for $j = 1, 2, \ldots$.

Therefore $\bigvee_{i=1}^{\infty} r_i$ is an upper bound for E. Also, since μ is a σ-additive probability measure, we conclude that

$$\mu\left(\bigvee_{i=1}^{\infty} r_i\right) = \lim_{j \to \infty} \mu\left(\bigvee_{i=1}^{j} r_i\right) = 0.$$

(ii) Let μ be an element of $\Omega_c(L) \cap \Omega_{JP}(L)$. By Zorn's lemma, there exists a maximal orthogonal set D in $\ker \mu$; set $p := \bigvee D$. Then

$$\mu(p) = \lim \left(\mu\left(\bigvee C\right)\right)_{C \in D^f} = 0,$$

hence p is an element of $\ker \mu$.

Now let r be an element of $\ker \mu$. Then there exists an element s in $\ker \mu$ which majorizes both elements p and r. By orthomodularity, we get

$$0 = \mu(s) = \mu(p) + \mu(p' \wedge s) = \mu(p' \wedge s).$$

But $p' \wedge s \perp D$ and by maximality of D in $\ker \mu$ we conclude that $p' \wedge s$ is the least element. Therefore p coincides with s which is an upper bound for r.

(iii) Let μ be an element of $\Omega_\sigma(L) \cap \Omega_{c\text{-}JP}(L)$ and let E be a maximal orthogonal set in L. Then the set

$$F := \{p \in E \colon \mu(p) \neq 0\}$$

is countable.

Assume $(L, \leqslant, ')$ to be an orthocomplete orthomodular poset and define

$$q := \bigvee (E - F).$$

Again by orthomodularity, we then conclude that q' is the supremum of F.

Since μ is an element of $\Omega_{c\text{-}JP}(L)$ the set $\ker \mu$ contains an upper bound for $E - F$. Therefore q is an element of $\ker \mu$ and

$$1 = \mu(q) + \mu(q') = \lim \left(\mu\left(\bigvee C\right)\right)_{C \in F^f} \leqslant \sup_{D \in E^f} \mu\left(\bigvee D\right)$$
$$= \lim \left(\mu\left(\bigvee D\right)\right)_{D \in E^f} \leqslant 1.$$

Hence μ is a completely additive measure.

Suppose now that $(L, \leqslant, ')$ is a σ-complete orthomodular lattice and set

$$r := \bigvee F.$$

Clearly, r' is an upper bound of the set $E - F$ and for any upper bound s of $E - F$ we then have

$$E - F \perp r' \wedge (r \vee s') \perp r$$

Therefore $E \perp r' \wedge (r \vee s')$ and by maximality of E we get

$$r' \wedge (r \vee s') = 0$$

or equivalently,

$$r' \leqslant s.$$

To this end we have shown that r' is the supremum of the set $E - F$. But μ is an element of $\Omega_{c\text{-}JP}(L)$, hence r' is an element of $\ker \mu$. The conclusion is now reached by precisely the same arguments as above.

3. The Positive Lebesgue Decomposition

Let $(L, \leqslant, ')$ be an orthomodular poset and, again, let $K(L)$ denote the cone of positive measures on L. As in classical measure theory we define the pair (λ, μ) of elements of $K(L)$ to be *singular*, denoted by $\lambda \perp \mu$, if there exists an element p in L such that

$$\lambda(p) = 0 = \mu(p').$$

An element λ of $K(L)$ is said to be *absolutely continuous* with respect to an element μ of $K(L)$, denoted by $\lambda \ll \mu$, provided that

$$\mu(p) = 0 \quad \text{implies that} \quad \lambda(p) = 0.$$

Let Δ be a convex subset of the convex set $\Omega(L)$ of probability measures on L. One verifies that the positive hull of Δ, denoted by $K(\Delta)$, is a subcone of $K(L)$. If Δ is non-empty then $K(\Delta)=\mathbb{R}_+\cdot\Delta$. We say that Δ has the *positive Lebesgue decomposition property* (PLD) provided that for each pair (λ, μ) of elements of $K(\Delta)$ there exists a pair (ν, ξ) of elements of $K(\Delta)$ such that

$$\text{(i)} \qquad \nu\perp\lambda, \qquad \xi\ll\lambda$$

and

$$\text{(ii)} \qquad \mu=\nu+\xi$$

For each element p in L we denote with $a_\Delta(p)$ the subset $\{\mu\in\Delta\colon \mu(p)=1\}$; notice that $a_\Delta(p)$ is a face of Δ. We are now in a position to give a "geometrical form" of the positive Lebesgue decomposition property.

LEMMA 3.1. *Let $(L, \leqslant, ')$ be an orthomodular poset and $\Omega(L)$ the convex set of probability measures on L. Let Δ be a convex subset of $\Omega(L)$.*

Then Δ has the positive Lebesgue decomposition property if and only if for each element λ of Δ the set

$$\bigcup_{q\in\ker\lambda}\operatorname{conv}\left(\bigcap_{p\in\ker\lambda}a_\Delta(p')\cup a_\Delta(q)\right)$$

coincides with Δ.

Proof. Suppose that Δ has the PLD. Let λ, μ be elements of Δ and let (ν, ξ) be a pair of elements in $K(\Delta)$ establishing the desired decomposition with respect to the pair (λ, μ).

We first assume that ν and ξ are different from 0. Then

$$\nu(1),\ \xi(1)>0 \quad\text{and}\quad \nu(1)+\xi(1)=1$$

and, moreover, the elements $\nu/\nu(1)$, $\xi/\xi(1)$ belong to Δ. We then have the following representation

$$\mu=\nu(1)(\nu/\nu(1))+\xi(1)(\xi/\xi(1)).$$

Clearly, for an element p in $\ker\lambda$ we get

$$(\xi/\xi(1))(p')=1.$$

Thus $\xi/\xi(1)$ is a member of the set $\bigcap_{p\in\ker\lambda}a_\Delta(p')$. Also, there exists an element q in L such that

$$\nu(q)=0=\lambda(q').$$

Thus q' is a member of $\ker\lambda$ and $\nu/\nu(1)$ belongs to $a_\Delta(q')$. The cases where ν or ξ are equal to 0 are readily dealt with. This proves necessity of the given condition.

The converse is straightforward.

As an immediate consequence, we note that the positive Lebesgue decomposition property is facially descendingly hereditary.

COROLLARY 3.2. *Let $(L, \leqslant, ')$ be an orthomodular poset and let Δ be a convex subset of $\Omega(L)$. If Δ has the positive Lebesgue decomposition property, then so has each of its faces.*

Proof. Let Δ' be a face of Δ. Then

$$\Delta' = \Delta' \cap \Delta = \bigcup_{q \in \ker\mu} \operatorname{conv}\Bigl(\bigcap_{p \in \ker\mu} (\Delta' \cap a_\Delta(p')) \cup (\Delta' \cap a_\Delta(q))\Bigr),$$

equality holding since

$$\bigcap_{p \in \ker\mu} a_\Delta(p') \quad \text{and} \quad a_\Delta(q), \qquad q \in \ker\mu,$$

are convex subsets of Δ. Now for any element r of L the set $\Delta' \cap a_\Delta(r)$ coincides with $a_{\Delta'}(r)$.

The *center* of an orthomodular poset $(L, \leqslant, ')$ is defined by

$$C(L) = \{p \in L\colon\ pCq \text{ for all } q \in L\}.$$

For a probability measure μ and an element p of the center $C(L)$ of L such that $\mu(p)$ is different from 0, the element $\mu_p \in \mathbb{R}^L$ defined by

$$\mu_p(q) = \mu(p \wedge q)/\mu(p), \qquad q \in L,$$

is again a probability measure. A subset Δ of $\Omega(L)$ is said to be *closed under central conditioning* provided that μ_p belongs to Δ for every element μ of Δ and every element p of $C(L)$ for which $\mu(p)$ is different from 0 and 1. We have the following

LEMMA 3.3. *Let $(L, \leqslant, ')$ be an orthomodular poset and $\Omega(L)$ the convex set of probability measures on L. Every face of $\Omega(L)$ is closed under central conditioning.*

Proof. Let Δ be a face of $\Omega(L)$. Let μ be an element of Δ, p an element of $C(L)$ and suppose that $\mu(p)$ is different from 0 and 1. Now μ_p and $\mu_{p'}$ belong to $\Omega(L)$ and for each element q in L we have

$$\begin{aligned}\mu(q) &= \mu((q \wedge p) \vee (q \wedge p')) = \mu(q \wedge p) + \mu(q \wedge p') \\ &= (\mu(p)\mu_p + (1 - \mu(p))\mu_{p'})(q).\end{aligned}$$

By the very definition of a face, we conclude that μ_p and $\mu_{p'}$ belong to Δ.

Remark. As a consequence, the sets $\Omega_\sigma(L)$ and $\Omega_c(L)$ are closed under central conditioning for any orthomodular poset $(L, \leqslant, ')$.

We now turn our attention to the Lebesgue decomposition in the classical context.

THEOREM 3.4. *Let $(L, \leqslant, ')$ be a Boolean orthomodular poset and let Δ be a convex subset of $\Omega(L)$.*

If Δ is a subset of $\Omega_{\sigma\text{-}JP}(L)$ and if Δ is closed under central conditioning then Δ has the positive Lebesgue decomposition property.

Remark. Let B be a field of subsets of a set X. Then the triple $(B, \subseteq, {}^c)$, where $\subseteq$ denotes set-inclusion and ^{c} denotes set-complementation, is a σ-complete Boolean orthomodular lattice. The classical theorem on the positive Lebesgue decomposition [7] is then recognized in Theorem 3.4. by equating Δ with $\Omega_\sigma(B)$.

Proof of Theorem 3.4. Let λ and μ be elements in Δ and let $(p_i)_{i=1}^\infty$ be a sequence in $\ker \lambda$ such that

$$\lim_{i\to\infty} \mu(p_i) = \sup\{\mu(p)\colon\ p \in \ker \lambda\}.$$

Since λ is an element of $\Omega_{\sigma\text{-}JP}(L)$ there exists in $\ker \lambda$ an upper bound of the set $\{p_i\colon\ i \in \mathbb{N}\}$, say q. Therefore μ attains the supremum on $\ker \lambda$ at q. We assume that $\mu(q)$ is different from 0 and 1. Then μ_q and $\mu_{q'}$ belong to Δ and

$$\mu = \mu(q)\mu_q + \mu(q')\mu_{q'}$$

since $(L, \leqslant, ')$ is Boolean.

Clearly, μ_q is a member of $a_\Delta(q)$. If r is any element of $\ker \lambda$, so is $q \vee (r \wedge q')$. Then

$$\mu(q) \geqslant \mu(q \vee (r \wedge q')) = \mu(q) + \mu(r \wedge q') \geqslant \mu(q),$$

thus $\mu(r \wedge q') = 0$. Therefore $\mu_{q'}$ is an element of $\bigcap_{p \in \ker \lambda} a_\Delta(p')$. The cases where $\mu(q)$ equals 0 or 1 are easily settled.

Let $(L, \leqslant, ')$ be an orthomodular poset. A subset Δ of $\Omega(L)$ is said to be *unital* for L if for each non-zero element p of L there exists an element μ in Δ such that

$$\mu(p) = 1.$$

We now proceed to the main result of this paragraph. In an obvious sense it represents a converse to the previous theorem. Applications to operator algebras are considered in the paragraph after next.

THEOREM 3.5. *Let $(L, \leqslant, ')$ be an orthocomplete orthomodular poset* *and let Δ be a unital and convex subset of $\Omega(L)$. If Δ is a subset of $\Omega_{c\text{-}JP}(L)$ and has the positive Lebesgue decomposition property then $(L, \leqslant, ')$ is Boolean.*

Proof. For any element λ in Δ denote with p_λ the orthocomplement of the largest element in $\ker \lambda$. We then have for r an element of L

$$\lambda(r) = 1 \Leftrightarrow p_\lambda \leqslant r;$$

notice that p_λ is different from 0.

Referring to Lemma 3.1., we get

$$\begin{aligned}\Delta &= \bigcup_{s \in \ker \lambda} \operatorname{conv}\left(\bigcap_{r \in \ker \lambda} a_\Delta(r') \cup a_\Delta(s)\right)\\ &= \operatorname{conv}\left(a_\Delta(p_\lambda) \cup a_\Delta(p'_\lambda)\right)\end{aligned}$$

since $a_\Delta(p_\lambda)$ coincides with $\bigcap_{r \in \ker \lambda} a_\Delta(r')$.

It follows that for elements μ and ν of Δ

$$a_\Delta(p_\mu) = \operatorname{conv}\{(a_\Delta(p_\mu) \cap a_\Delta(p_\nu)) \cup (a_\Delta(p_\mu) \cap a_\Delta(p'_\nu))\}$$

and

$$a_\Delta(p_\nu) = \operatorname{conv}\{(a_\Delta(p_\mu) \cap a_\Delta(p_\nu)) \cup (a_\Delta(p'_\mu) \cap a_\Delta(p_\nu))\}$$

holds true. Let u be a maximal lower bound of p_μ and p_ν, e.g. take the supremum of a maximal orthogonal set of lower bounds of p_μ and p_ν. Similarly, let v and w be maximal lower bounds of p_μ, p'_ν and p'_μ, p_ν, respectively. Then clearly,

$$u \perp v \perp w \perp u$$

and

$$u \vee v \leqslant p_\mu, \qquad u \vee w \leqslant p_\nu.$$

Assume now that p_μ is not equal to $u \vee v$. Then, by orthomodularity, there exists a non-zero element z of L such that

$$z \perp u \vee v \quad \text{and} \quad z \leqslant p_\mu.$$

Since Δ is unital there exists an element κ in Δ which evaluates to 1 on z and therefore belongs to $a_\Delta(p_\mu)$. If κ is neither an element of $a_\Delta(p_\mu) \cap a_\Delta(p_\nu)$ nor of $a_\Delta(p_\mu) \cap a_\Delta(p'_\nu)$ then there exist elements ξ and ψ in $a_\Delta(p_\mu) \cap a_\Delta(p_\nu)$ and in $a_\Delta(p_\mu) \cap a_\Delta(p'_\nu)$, respectively, and a real number t in the open unit interval of the reals such that

$$\kappa = t\xi + (1-t)\psi.$$

We then have

$$\xi(p_\kappa) = \psi(p_\kappa) = 1.$$

Hence

$$p_\xi \leqslant p_\kappa \leqslant z.$$

Now

$$p_\xi \perp u$$

and also

$$p_\xi \leqslant p_\mu, p_\nu,$$

thus

$$u < u \vee p_\xi \leqslant p_\mu, p_\nu,$$

a contradiction. The cases where κ belongs to $a_\Delta(p_\mu) \cap a_\Delta(p_\nu)$ or to $a_\Delta(p_\mu) \cap a(p'_\nu)$ are similarly contradictory. This proves that

$$p_\mu = u \vee v.$$

Corresponding arguments lead to

$$p_\nu = u \vee w.$$

To this end we have shown that $p_\mu C p_\nu$ for all elements μ and ν in Δ.

Let p be a non-zero element in L; then $a_\Delta(p)$ is not empty. For any maximal orthogonal subset D of $\{p_\mu\colon \mu \in a_\Delta(p)\}$ the element p is the supremum of D; this follows from orthomodularity, orthocompleteness and unitality. Now, for any element ν in Δ we have $p_\nu C D$ and, moreover, $\bigvee_{s \in D} (p_\nu \wedge s)$ exists. By a theorem of J. Pool [16, 17], it follows that p is compatible with p_ν. A repetition of this argument leads to the conclusion that any pair of elements of L is compatible.

4. Permanence Properties

The purpose of this paragraph is to show how the positive Lebesgue decomposition property behaves under the formation of direct sums and direct products of orthomodular posets.

Let $(L_i, \leqslant_i, {}'^i)_{i \in I}$ be a family of orthomodular posets indexed by the set I. Denote with 1_i and 0_i the greatest and the least element in $(L_i, \leqslant_i)$, respectively, for i an element of I. Let $\dot{\bigcup}_{i \in I} L_i$ be the set-theoretical sum (disjoint union) of the family $(L_i)_{i \in I}$ and let $j_i\colon L_i \to \dot{\bigcup}_{i \in I} L_i$, $i \in I$, be the injection maps.

For elements x and y in $\dot{\bigcup}_i L_i$ let p and q be elements in L_i and L_k, respectively, such that

$$x = j_i(p) \quad \text{and} \quad y = j_k(q).$$

Then the relation

$$x \preccurlyeq y :\Leftrightarrow i = k,\ p \leqslant_i q \quad \text{or} \quad i \neq k,\ p = 0_i \\ \text{or} \quad i \neq k,\ q = 1_k$$

defines a semi-order on $\bigcup_i L_i$. Let $\sum_{i \in I} L_i$ be the quotient of $\bigcup_i L_i$ with respect to the equivalence relation induced by the semi-order $\preccurlyeq$ and let $c\colon \bigcup_i L_i \to \sum_{i \in I} L_i$ be the canonical quotient map. Let u, v be elements of $\sum_{i \in I} L_i$ and x, y be elements of $\bigcup_i L_i$ such that

$$u = c(x) \quad \text{and} \quad v = c(y);$$

then the relation

$$u \leqslant v :\Leftrightarrow x \preccurlyeq y$$

is an order relation on $\sum_{i \in I} L_i$. Define for $i \in I$ the mapping $s_i\colon L_i \to \sum_{i \in I} L_i$ as the composition of j_i and c.

Clearly, $\sum_{i \in I} L_i$ coincides with the union of the subsets $s_i(L_i)$, $i \in I$. For p an element of L_i, q an element of L_k and the index i different from k we then have

$$s_i(p) \leqslant s_k(q) \quad \text{iff} \quad p = 0_i \quad \text{or} \quad q = 1_k$$

Also, if p and q are elements of L_i, $i \in I$, then

$$s_i(p) \leqslant s_i(q) \quad \text{if and only if} \quad p \leqslant_i q.$$

Let $(p_j)_j$ be a family in L_i. Then $\bigvee_j p_j$ exists if and only if $\bigvee_j s_i(p_j)$ exists; if either is the case then $s_i\left(\bigvee_j p_j\right)$ coincides with $\bigvee_j s_i(p_j)$. Denote with 1 the element $s_i(1_i)$ and with 0 the element $s_i(0_i)$, for $i \in I$. Clearly, 1 is the greatest element and 0 is the least element in $\left(\sum_{i \in I} L_i, \leqslant\right)$.

Let u be an element in $\sum_{i \in I} L_i$ and let p be an element in L_i such that u coincides with $s_i(p)$; denote with u' the element $s_i(p')$. Then the mapping $'\colon \sum_i L_i \to \sum_i L_i$ is an orthocomplementation on the poset $\left(\sum_i L_i, \leqslant\right)$ such that $\left(\sum_i L_i, \leqslant, '\right)$ becomes an orthomodular poset, the *direct sum* of the family $(L_i, \leqslant_i, '^i)_{i \in I}$.

For each $i \in I$ define the mapping s_i^*: $K\left(\sum_i L_i\right) \to \mathbb{R}^{L_i}$ by

$$s_i^*(\mu)(p) = \mu(s_i(p)),$$

where μ is an element of $K\left(\sum_i L_i\right)$ and p an element of L_i. One verifies that for each index i s_i^* maps $K\left(\sum_i L_i\right)$, resp. $K_\sigma\left(\sum_i L_i\right)$, resp. $K_c\left(\sum_i L_i\right)$, into the cone $K(L_i)$, resp. $K_\sigma(L_i)$, resp. $K_c(L_i)$, and that it preserves positive combinations of elements of $K\left(\sum_i L_i\right)$.

LEMMA 4.1. *Let λ and μ be elements of $K\left(\sum_i L_i\right)$. Then*

(i) $\lambda \perp \mu \Leftrightarrow$ *there exists $i \in I$ such that $s_i^*(\lambda) \perp s_i^*(\mu)$;*
(ii) $\lambda \ll \mu \Leftrightarrow$ *for all $i \in I$ $s_i^*(\lambda) \ll s_i^*(\mu)$.*

Proof. The proof is straightforward.

Let $\times_i K(L_i)$ be the set-theoretical product of the family $(K(L_i))_{i\in I}$ and let qr_i: $\times_i K(L_i) \to K(L_i)$, $i \in I$, be the projection maps. Define a mapping Ψ from $K\left(\sum_i L_i\right)$ into $\times_i K(L_i)$ by

$$qr_i(\Psi(\mu)) = s_i^*(\mu) \quad \text{for all} \quad i \in I.$$

Positive σ-additive and positive completely additive measures are treated accordingly.

LEMMA 4.2. *The mapping Ψ: $K\left(\sum_i L_i\right) \to \times_i K(L_i)$ is injective and its image is equal to the set $\left\{f \in \times_i K(L_i)\text{: } (qr_i f)(1_i) = (qr_j f)(1_j) \text{ for all } i, j \in I\right\}$. Analogous statements hold for $K_\sigma\left(\sum_i L_i\right)$ and $K_c\left(\sum_i L_i\right)$.*

Proof. A verification!

LEMMA 4.3. *The cone $K\left(\sum_i L_i\right)$, resp. $K_\sigma\left(\sum_i L_i\right)$, resp. $K_c\left(\sum_i L_i\right)$, consists of the zero measure alone if and only if there exists an index i such that $K(L_i)$, resp. $K_\sigma(L_i)$, resp. $K_c(L_i)$, consists of the zero measure alone.*

If either is not the case then for all indices i the mapping s_i^* maps

$K\left(\sum_i L_i\right)$, *resp.* $K_\sigma\left(\sum_i L_i\right)$, *resp.* $K_c\left(\sum_i L_i\right)$, *onto* $K(L_i)$, *resp.* $K_\sigma(L_i)$, *resp.* $K_c(L_i)$.

Proof. (i) Suppose that for all elements i in I the cone $K(L_i)$ is different from $\{0\}$. Select for each index i a non-zero element λ_i in $K(L_i)$ and define an element f in $\bigtimes_i K(L_i)$ by

$$qr_i(f) = \lambda_i/\lambda_i(1_i), \qquad i \in I.$$

Clearly, f is in the image of Ψ. If $\Psi^{-1}(f)$ is equal to zero then

$$\lambda_i/\lambda_i(1_i) = qr_i(f) = s_i^*(0) = 0,$$

a contradiction.

(ii) Suppose that $K\left(\sum_i L_i\right)$ has a non-zero element μ. Then $\mu(s_i(1_i))$ and therefore $s_i^*(\mu)(1_i)$ is unequal zero for all $i \in I$. This proves that for any index i $s_i^*(\mu)$ is not the zero measure.

(iii) Let λ_i be a non-zero element in $K(L_i)$. for each $j \in I$, $j \neq i$, select a non-zero element λ_j in $K(L_j)$ and define an element f in $\bigtimes_i K(L_i)$ by

$$qr_l(f) = \lambda_i(1_i)\lambda_l/\lambda_l(1_l), \qquad l \in I.$$

Then f is an element of $\Psi\left(K\left(\sum_i L_i\right)\right)$ and

$$s_i^*(\Psi^{-1}(f)) = qr_i(\Psi\Psi^{-1}(f)) = \lambda_i.$$

The proofs of the remaining cases are analogous.

THEOREM 4.4. *Let* $(L_i, \leqslant_i, {}'^i)_{i\in I}$ *be a family of orthomodular posets and let* $\left(\sum_i L_i, \leqslant, '\right)$ *be their direct sum. Suppose that* $\Omega(L_i)$, *resp.* $\Omega_\sigma(L_i)$, *resp.* $\Omega_c(L_i)$, *is not empty for all* $i \in I$.

If $\Omega\left(\sum_i L_i\right)$, *resp.* $\Omega_\sigma\left(\sum_i L_i\right)$, *resp.* $\Omega_c\left(\sum_i L_i\right)$, *has the positive Lebesgue decomposition property, then so has* $\Omega(L_i)$ *resp.* $\Omega_\sigma(L_i)$, *resp.* $\Omega_c(L_i)$, *for all* $i \in I$.

Proof. Suppose that $\Omega\left(\sum_i L_i\right)$ has the PLD and is not empty. Let λ, μ be elements of $K(L_i)$; we may assume that λ is unequal zero. For each index j different from i select an element λ_j in $K(L_j)$ such that

$$\lambda_j(1_j) = \lambda(1_i) \quad \text{for all} \quad j \neq i$$

which is possible by Lemma 4.3. Set

$$\mu_j := \mu(1_i)\lambda_j/\lambda(1_i) \quad \text{for} \quad j \neq i$$

and define elements f and g in $\bigtimes_i K(L_i)$ by

$$qr_i(f) = \lambda, \qquad qr_j(f) = \lambda_j \quad \text{for} \quad j \neq i,$$
$$qr_i(g) = \mu, \qquad qr_j(g) = \mu_j \quad \text{for} \quad j \neq i.$$

Obviously, f and g belong to the image of Ψ.

Let ξ and ν be elements of $K\left(\sum_i L_i\right)$ such that

$$\Psi^{-1}(f) \perp \xi, \qquad \Psi^{-1}(f) \gg \nu \quad \text{and} \quad \Psi^{-1}(g) = \xi + \nu.$$

By Lemma 4.1., there exists an index l such that

$$s_l^*(\Psi^{-1}(f)) \perp_l s_l^*(\xi).$$

If $l = i$ we conclude that

$$\lambda = qr_i(\Psi\Psi^{-1}(f)) = s_i^*(\Psi^{-1}(f)) \perp_i s_i^*(\xi),$$
$$\lambda = s_i^*(\Psi^{-1}(f)) \gg s_i^*(\nu),$$
$$\mu = qr_i(\Psi\Psi^{-1}(g)) = s_i^*(\Psi^{-1}(g)) = s_i^*(\xi) + s_i^*(\nu)$$

and we are done. Suppose now that $l \neq i$ and let p be an element of L_l such that

$$\lambda_l(p) = \Psi^{-1}(f)(s_l(p)) = s_l^*(\Psi^{-1}(f))(p) = 0 = s_l^*(\xi)(p').$$

Then

$$0 = \lambda_l(p) = \mu_l(p) = \Psi^{-1}(g)(s_l(p)) = \xi(s_l(p)) + \nu(s_l(p))$$
$$= \xi(s_l(p)).$$

Therefore ξ is equal to zero and $\Psi^{-1}(f) \gg \Psi^{-1}(g)$. Then clearly

$$\lambda \perp 0, \quad \lambda \gg \mu \quad \text{and} \quad \mu = \mu + 0.$$

The proofs for the positive σ-additive measures as well as the positive completely additive measures are similar.

There are examples of direct sums $\sum_i L_i$ with infinitely many summands L_i different from $\{0_i, 1_i\}$ and $\Omega\left(\sum_i L_i\right)$ having the positive Lebesgue decomposition property. However we have:

THEOREM 4.5. *Let $(L_i, \leq_i, '^i)_{i \in I}$ be a family of orthomodular posets indexed by the set I. Suppose that L_i is different from $\{0_i, 1_i\}$ and that*

$\Omega(L_i)$, resp. $\Omega_\sigma(L_i)$, resp. $\Omega_c(L_i)$, is unital for all $i \in I$. If $\Omega(\sum_i L_i)$, resp. $\Omega_\sigma(\sum_i L_i)$, resp. $\Omega_c(\sum_i L_i)$, has the positive Lebesgue decomposition property then the indexing set I is finite.

Proof. Assume that $\Omega\left(\sum_i L_i\right)$ has the PLD and that I is infinite. Let I' be a countably infinite subset of I and let $n \in \mathbb{N} \to i(n) \in I'$ be a strict enumeration of I'. For each natural number n select an element p_n in $L_{i(n)} - \{0_{i(n)}, 1_{i(n)}\}$ and elements λ_n, μ_n in $\Omega(L_{i(n)})$ such that

$$\lambda_n(p_n) = 1 = \mu_n(p_n').$$

Define elements f and g in $\bigtimes_i K(L_i)$ by

$$qr_{i(n)} = \lambda_n \quad \text{for} \quad n \in \mathbb{N} \quad \text{and} \quad qr_j(f) = \lambda_j \quad \text{for} \quad j \in I - I',$$

where λ_j is an arbitrary element of $\Omega(L_j)$;

$$qr_{i(n)}(g) = (1/n)\lambda_n + (1 - 1/n)\mu_n \quad \text{for} \quad n \in \mathbb{N}$$

and

$$qr_j(g) = \lambda_j \quad \text{for} \quad j \in I - I'.$$

Clearly, f and g belong to the image of Ψ.

Let ξ and ν be elements of the cone $K\left(\sum_i L_i\right)$ such that

$$\Psi^{-1}(f) \perp \nu, \qquad \Psi^{-1}(f) \gg \xi \quad \text{and} \quad \Psi^{-1}(g) = \nu + \xi.$$

For all $n \in \mathbb{N}$ $\Psi^{-1}(f)(s_{i(n)}(p_n'))$ is equal to zero, hence

$$\xi(s_{i(n)}(p_n')) = 0, \qquad n \in \mathbb{N}.$$

Now, for all $n \in \mathbb{N}$,

$$\begin{aligned} \xi(1) &= \xi(s_{i(n)}(p_n')) + \xi(s_{i(n)}(p_n)) = \xi(s_{i(n)}(p_n)) \\ &\leq \Psi^{-1}(g)(s_{i(n)}(p_n)) = 1/n, \end{aligned}$$

thus ξ is the zero measure and therefore $\Psi^{-1}(f) \perp \Psi^{-1}(g)$. By Lemma 4.1 there exists an index j in I and an element q in L_j such that

$$s_j^*(\Psi^{-1}(f))(q) = 0 = s_j^*(\Psi^{-1}(g))(q').$$

If j is an element of $I - I'$ then

$$\lambda_j(q) = 0 = \lambda_j(q')$$

hence $\lambda_j(1_j) = 0$. If $j = i(n)$ for some $n \in \mathbb{N}$ then

$$\lambda_n(q) = 0 = (1/n)\lambda_n(q') + (1 - 1/n)\mu_n(q'),$$

hence $\lambda_n(q') = 0$, thus $\lambda_n(1_{i(n)}) = 0$. Therefore in both cases we get a contradiction. The balance of the assertion is taken care of in a similar way.

THEOREM 4.6. *Let* $(L_i, \leqslant_i, '^i)_{i\in I}$ *be a finite family of orthomodular posets.*

If $\Omega(L_i)$, *resp.* $\Omega_\sigma(L_i)$, *resp.* $\Omega_c(L_i)$, *has the positive Lebesgue decomposition property for all* $i \in I$, *then so has* $\Omega\left(\sum_i L_i\right)$, *resp.* $\Omega_\sigma\left(\sum_i L_i\right)$, *resp.* $\Omega_c\left(\sum_i L_i\right)$.

Proof. Suppose that $\Omega(L_i)$ has the PLD for all $i \in I$. We may assume that $\Omega\left(\sum_i L_i\right)$ is not empty. Let λ, μ be elements of $K\left(\sum_i L_i\right)$. Then for each index i there exist elements ξ_i, ν_i of $K(L_i)$ such that

$$s_i^*(\lambda) \perp_i \xi_i, \qquad s_i^*(\lambda) \gg \nu_i \quad \text{and} \quad s_i^*(\mu) = \xi_i + \nu_i.$$

Let k be an index such that

$$\nu_k(1_k) \leqslant \nu_i(1_i) \quad \text{for all} \quad i \in I.$$

If $\nu_k(1_k)$ is strictly greater than zero we define an element f in $\mathop{\times}\limits_i K(L_i)$ through

$$qr_i(f) = \nu_k(1_k)\nu_i/\nu_i(1_i), \qquad i \in I.$$

Then f is in the image of Ψ.

For any element i in I and any element p in L_i we have

$$\begin{aligned}(\mu - \Psi^{-1}(f))(s_i(p)) &= s_i^*(\mu)(p) - \nu_k(1_k)\nu_i(p)/\nu_i(1_i)\\ &\geqslant s_i^*(\mu)(p) - \nu_i(p) = \xi_i(p) \geqslant 0\end{aligned}$$

Hence $\mu - \Psi^{-1}(f)$ is an element of $K\left(\sum_i L_i\right)$. Moreover, $\lambda \perp (\mu - \Psi^{-1}(f))$ since

$$s_k^*(\mu - \Psi^{-1}(f)) = s_k^*(\mu) - \nu_k(1_k)\nu_k/\nu_k(1_k) = \xi_k$$

and $\xi_k \perp s_k^*(\lambda)$. Also, $\lambda \gg \Psi^{-1}(f)$ since $s_i^*(\lambda) \gg \nu_i$ implies that

$$s_i^*(\lambda) \gg \nu_k(1_k)\nu_i/\nu_i(1_i) = s_i^*(\Psi^{-1}(f)), \quad \text{for all} \quad i \in I.$$

If $\nu_k(1_k)$ equals zero, then ν_k is equal to zero and $(s_k^*(\mu), s_k^*(\lambda))$ forms a singular pair. Therefore $\mu \perp \lambda$. The proof for the σ-additive measures as well as the completely additive measures is similar.

We now turn our attention to direct products of orthomodular posets. Let $(L_i, \leqslant_i, '^i)_{i\in I}$ be a family of orthomodular posets indexed by the set I.

Let $\bigtimes_{i\in I} L_i$ be the set-theoretical product (Cartesian product) of the family $(L_i)_i$ and let pr_i: $\bigtimes_i L_i \to L_i$, $i \in I$, be the projection maps.

Let u, v be elements of $\bigtimes_i L_i$; define an order relation in $\bigtimes_i L_i$ by

$$u \leq v \;:\Leftrightarrow pr_i(u) \leq_i pr_i(v) \quad \text{for all} \quad i \in I.$$

Then the element 1, resp. 0, defined by

$$pr_i(1) = 1_i, \text{ resp. } pr_i(0) = 0_i, \quad \text{for all} \quad i \in I,$$

is the greatest, resp. the least, element in the poset $\left(\bigtimes_i L_i, \leq\right)$. For any element u in $\bigtimes_i L_i$ define an element u' in $\bigtimes_i L_i$ through

$$pr_i(u') = (pr_i(u))'^{i} \quad \text{for all} \quad i \in I.$$

Then the mapping $'$: $\bigtimes_i L_i \to \bigtimes_i L_i$ is an orthocomplementation which makes the triple $\left(\bigtimes_i L_i, \leq, '\right)$ into an orthomodular poset, the *direct product* of the family $(L_i, \leq_i, '^i)_{i\in I}$.

For each index i the mapping pr_i is surjective and preserves order as well as orthocomplementation. Let $(u_k)_k$ be a family of elements of $\bigtimes_i L_i$. Then $\bigvee_k u_k$ exists if and only if $\bigvee_k pr_i(u_k)$ exists for all $i \in I$; if either is the case then $pr_i\left(\bigvee_k u_k\right)$ coincides with $\bigvee_k pr_i(u_k)$ for all $i \in I$.

Define a mapping e_i: $L_i \to \bigtimes_i L_i$, for all $i \in I$, through

$$pr_j(e_i(p)) = \begin{cases} p, & \text{if} \quad i = j, \\ 0_j, & \text{if} \quad i \neq j. \end{cases}$$

Clearly, for each index i the mapping e_i is injective and preserves orthogonality. Also, for elements p and q in L_i

$$p \leq_i q \Leftrightarrow e_i(p) \leq e_i(q).$$

Let $(p_k)_k$ be a family of elements in L_i. Then $\bigvee_k p_k$ exists if and only if $\bigvee_k e_i(p_k)$ exists; if either is the case then $e_i\left(\bigvee_k p_k\right)$ coincides with $\bigvee_k e_i(p_k)$. Let I' be a non-empty subset of I and select for each $i \in I$ an element p_i of L_i. Then $\{e_i(p_i)\colon i \in I'\}$ is an orthogonal set and the element u

in $\mathop{\times}_i L_i$ defined by

$$pr_i(u) = \begin{cases} p_i, & i \in I' \\ 0_i, & i \notin I' \end{cases}$$

is the supremum of this set. Notice that $\{e_i(1_i)\colon i \in I\}$ is a maximal orthogonal set in $\mathop{\times}_i L_i - \{0\}$.

For each $i \in I$ define mappings $pr_i^*\colon K(L_i) \to \mathbb{R}^{\times_i L_i}$ and $e_i^*\colon K\left(\mathop{\times}_i L_i\right) \to \mathbb{R}^{L_i}$ by

$$pr_i^*(\lambda)(u) = \lambda(pr_i(u)), \qquad \lambda \in K(L_i), \qquad u \in \mathop{\times}_i L_i$$

and

$$e_i^*(\mu)(p) = \mu(e_i(p)), \qquad \mu \in K\left(\mathop{\times}_i L_i\right), \qquad p \in L_i.$$

One verifies that the image of $K(L_i)$, resp. $K_\sigma(L_i)$, resp. $K_c(L_i)$, under the mapping pr_i^* is contained in $K\left(\mathop{\times}_i L_i\right)$, resp. $K_\sigma\left(\mathop{\times}_i L_i\right)$, resp. $K_c\left(\mathop{\times}_i L_i\right)$. Similarly, the mapping e_i^* sends the cone $K\left(\mathop{\times}_i L_i\right)$, resp. $K_\sigma\left(\mathop{\times}_i L_i\right)$, resp. $K_c\left(\mathop{\times}_i L_i\right)$, into $K(L_i)$, resp. $K_\sigma(L_i)$, resp. $K_c(L_i)$, for all $i \in I$. Both mappings preserve positive combinations of positive measures and the composition of pr_i^* with e_i^* equals the identity mapping on $K(L_i)$ for all $i \in I$.

LEMMA 4.7. (i) *Let λ, μ be elements in $K\left(\mathop{\times}_i L_i\right)$. Then $\lambda \ll \mu$ implies $e_i^*(\lambda) \ll e_i^*(\mu)$ and $\lambda \perp \mu$ implies $e_i^*(\lambda) \perp e_i^*(\mu)$ for all $i \in I$.*

(ii) *Let ξ, ν be elements in $K(L_i)$. Then $\xi \ll \nu$ implies $pr_i^*(\xi) \ll pr_i^*(\nu)$ and $\xi \perp \nu$ implies $pr_i^*(\xi) \perp pr_i^*(\nu)$ for all $i \in I$.*

Proof. (i) Let λ, μ be elements in $K\left(\mathop{\times}_i L_i\right)$ with $\lambda \perp \mu$. Then there exists an element u in $\mathop{\times}_i L_i$ such that

$$\lambda(u) = 0 = \mu(u').$$

Since

$$(e_i \circ pr_i)(u) \leqslant u \quad \text{and} \quad (e_i \circ pr_i)(u') \leqslant u'$$

we conclude that

$$0 = \lambda((e_i \circ pr_i)(u)) = e_i^*(\lambda)(pr_i(u))$$

and similarly

$$0 = e_i^*(\mu)(pr_i(u')) = e_i^*(\mu)(pr_i(u)').$$

(ii) Let ξ and ν be elements in $K(L_i)$ with $\xi \perp \nu$; then there exists an element p in L_i such that

$$\xi(p) = 0 = \nu(p').$$

Clearly, $pr_i^*(\xi)(e_i(p)) = 0$. By orthomodularity, the orthocomplement of $e_i(p)$ is equal to $e_i(p') \vee \bigvee_{j \neq i, j \in I} e_j(1_j)$ and we have

$$pr_i^*(\nu)\Big(e_i(p') \vee \bigvee_{j \neq i} e_j(1_j)\Big) = \nu((pr_i \circ e_i)(p')) + \nu\Big(pr_i\Big(\bigvee_{j \neq i} e_j(1_j)\Big)\Big) = \nu(p') + \nu(0_i) = 0.$$

THEOREM 4.8. *Let $(L_i, \leqslant_i, {}'^i)_{i \in I}$ be a family of orthomodular posets and let $\left(\bigtimes_i L_i, \leqslant, {}'\right)$ be their direct product.*

If $\Omega\left(\bigtimes_i L_i\right)$, resp. $\Omega_\sigma\left(\bigtimes_i L_i\right)$, resp. $\Omega_c\left(\bigtimes_i L_i\right)$, has the positive Lebesgue decomposition property, then so has $\Omega(L_i)$, resp. $\Omega_\sigma(L_i)$, resp. $\Omega_c(L_i)$, for all $i \in I$.

Proof. This is an immediate consequence of Lemma 4.7 and the previous remarks.

Let $(i(n))_{n \in \mathbb{N}}$ be a sequence in the indexing set I. For each natural number n let μ_n be an element of $K(L_{i(n)})$, resp. $K_\sigma(L_{i(n)})$, resp. $K_c(L_{i(n)})$, such that

$$\sum_{n=1}^{m} \mu_n(1_{i(n)}) \leqslant M \quad \text{for all} \quad m \in \mathbb{N}$$

and some positive scalar M. Then the τ-limit of the sequence

$$\left(\sum_{n=1}^{m} pr_{i(n)}^*(\mu_n)\right)_{m \in \mathbb{N}}$$

exists and belongs to $K\left(\bigtimes_i L_i\right)$, resp. $K_\sigma\left(\bigtimes_i L_i\right)$, resp. $K_c\left(\bigtimes_i L_i\right)$, [**19**, thm. 2.2.]. Conversely we have

THEOREM 4.9. *Let I be a non-empty set, resp. a non-empty countable set. Let $(L_i, \leqslant_i, {}'^i)_{i \in I}$ be a family of orthomodular posets indexed by the set I. Then for each element μ in $K_c\left(\bigtimes_i L_i\right)$, resp. $K_\sigma\left(\bigtimes_i L_i\right)$, there exists a* *sequence* $(i(n))_{n\in\mathbb{N}}$ *in* I *and for each natural number* n *there exists an element* μ_n *in* $K_c(L_{i(n)})$, *resp.* $K_\sigma(L_{i(n)})$, *such that*

$$\mu = \tau - \lim_{m\to\infty} \sum_{n=1}^{m} pr_{i(n)}^{*}(\mu_n).$$

If I *is a non-empty finite set and* μ *an element of* $K(\bigtimes_i L_i)$, *then for each* $i \in I$ *there exists an element* μ_i *in* $K(L_i)$ *such that*

$$\mu = \sum_{i\in I} pr_i^{*}(\mu_i).$$

Proof. We only consider the case where I is an infinite set and μ a non-zero element of $K_c\left(\bigtimes_i L_i\right)$. Since $\{e_i(1_i)\colon i \in I\}$ is a maximal orthogonal set in $\bigtimes_i L_i - \{0\}$ we conclude that the subset

$$F := \{i \in I\colon \ \mu(e_i(1_i)) \neq 0\}$$

is non-empty and countable. We may assume that F is infinite. Let $n \in \mathbb{N} \to i(n) \in F$ be a strict enumeration of F. Since

$$\sum_{n=1}^{m} e_{i(n)}^{*}(\mu)(1_{i(n)}) = \sum_{n=1}^{m} \mu(e_{i(n)}(1_{i(n)})) \leq \mu(1)$$

we conclude that

$$\tau - \lim_{m\to\infty} \sum_{n=1}^{m} pr_{i(n)}^{*}(e_{i(n)}^{*}(\mu))$$

exists and belongs to $K_c\left(\bigtimes_i L_i\right)$.

It then follows for u an element of $\bigtimes_i L_i$ that

$$\begin{aligned}
\lim_{m\to\infty} \sum_{n=1}^{m} pr_{i(n)}^{*}(e_{i(n)}^{*}(\mu))(u) &= \lim_{m\to\infty} \mu\left(\bigvee_{n=1}^{m} (e_{i(n)} \circ pr_{i(n)})(u)\right) \\
&= \mu\left(\bigvee_{n=1}^{\infty} (e_{i(n)} \circ pr_{i(n)})(u)\right) + \mu\left(\bigvee_{i\notin F} (e_i \circ pr_i)(u)\right) \\
&= \mu\left(\bigvee_{i\in I} (e_i \circ pr_i)(u)\right) = \mu(u)
\end{aligned}$$

since μ is completely additive.

Remark. The representation of a positive σ-additive measure on the direct product as given in the first part of Theorem 4.9 is also valid if I is a non-real-measurable set [**14**].

THEOREM 4.10. *Let I be a non-empty set, resp. a non-empty countable set, resp. a non-empty finite set. Let $(L_\iota, \leqslant_\iota, {}'^\iota)_{\iota\in I}$ be a family of orthomodular posets indexed by the set I and let $\left(\bigtimes_i L_i, \leqslant, '\right)$ be their direct product.*

If $\Omega_c(L_i)$, resp. $\Omega_\sigma(L_i)$, resp. $\Omega(L_i)$, has the positive Lebesgue decomposition property for all $i \in I$, then so has $\Omega_c\left(\bigtimes_i L_i\right)$, resp. $\Omega_\sigma\left(\bigtimes_i L_i\right)$, resp. $\Omega\left(\bigtimes_i L_i\right)$.

Proof. Suppose that I is an infinite set and that $\Omega_c(L_i)$ has the PLD for all $i \in I$. Let λ, μ be elements of $K_c\left(\bigtimes_i L_\iota\right)$. Then, by Theorem 4.8 and after the obvious manipulations, there exists a sequence $(i(n))_{n\in\mathbb{N}}$ in I and to each $n \in \mathbb{N}$ there exist elements λ_n, μ_n in $K_c(L_{i(n)})$ such that

$$\lambda = \tau - \lim_{m\to\infty} \sum_{n=1}^{m} pr^*_{i(n)}(\lambda_n)$$

and

$$\mu = \tau - \lim_{m\to\infty} \sum_{n=1}^{m} pr^*_{i(n)}(\mu_n).$$

To each natural number n let ν_n, ξ_n be elements in $K_c(L_{i(n)})$ such that

$$\lambda_n \perp \nu_n, \qquad \lambda_n \gg \xi_n \quad \text{and} \quad \mu_n = \nu_n + \xi_n$$

and define ν and ξ as the τ-limits of the sequences $\left(\sum_{n=1}^{m} pr^*_{i(n)}(\nu_n)\right)_{m\in\mathbb{N}}$ and $\left(\sum_{n=1}^{m} pr^*_{i(n)}(\xi_n)\right)_{m\in\mathbb{N}}$, respectively. The elements ν and ξ belong to the cone $K_c\left(\bigtimes_i L_i\right)$ and μ coincides with their sum. If for an element u in $\bigtimes_i L_i \lambda(u)$ equals zero then

$$0 = pr^*_{i(n)}(\lambda_n)(u) = \lambda_n(pr_{i(n)}(u)) \quad \text{for all} \quad n \in \mathbb{N}.$$

Then

$$0 = \xi_n(pr_{i(n)}(u)) = pr^*_{i(n)}(\xi_n)(u)$$

which implies that $\xi(u)$ coincides with zero.

For each natural number n select an element p_n in $L_{i(n)}$ such that

$$\lambda_n(p_n) = 0 = \nu_n(p'_n).$$

Define

$$u := \bigvee_{n\in\mathbb{N}} e_{i(n)}(p_n).$$

Then, by orthomodularity,

$$u' = \bigvee_{n \in \mathbb{N}} e_{i(n)}(p_n') \vee \bigvee_{j \neq i(n)} e_j(1_j).$$

Since λ and ν are completely additive measures we conclude that

$$\lambda(u) = 0 = \nu(u').$$

An orthomodular poset is said to be *constructible* **[11]** if it can be constructed from a family of Boolean orthomodular lattices by means of a finite number of applications of the direct sum and the direct product operations. Among others we get the following corollary

COROLLARY 4.11. *Let $(L, \leqslant, ')$ be a finite constructible orthomodular poset. Then $\Omega(L)$ has the positive Lebesgue decomposition property.*

Proof. Note that if $(L, \leqslant, ')$ is a finite Boolean orthomodular poset, then $\Omega(L)$ equals $\Omega_{\sigma\text{-}JP}(L)$ and therefore, by Theorem 3.4, $\Omega(L)$ has the PLD.

5. Applications to JBW-Algebras

We study the main results of §3 in the context of JBW-algebras.

A real algebra A, not necessarily associative, for which

$$a \circ b = b \circ a,$$
$$a \circ (b \circ a^2) = (a \circ b) \circ a^2$$

holds true and which is also a Banach space with respect to a norm $\| \ \|: A \to \mathbb{R}_+$ satisfying

$$\|a \circ b\| \leqslant \|a\| \cdot \|b\|,$$
$$\|a^2\| = \|a\|^2$$

and

$$\|a^2\| \leqslant \|a^2 + b^2\|$$

is said to be a JB-*algebra*.

An element a in A is called *positive* if there exists an element b such that

$$a = b \circ b.$$

The set A_+ consisting of the positive elements in A forms a generating cone in A. An *idempotent* is an element p in A satisfying

$$p = p \circ p;$$

$U(A)$ denotes the collection of idempotents of A.

A JB-algebra which is the Banach dual space of a, necessarily unique, Banach space is called a JBW-*algebra*. A JBW-algebra has a unit, denoted by e.

Let A be a JBW-algebra with predual Banach space A_*. For each element a in A, the weak*-continuous linear mapping U_a: $A \to A$ is defined by

$$U_a b = \{aba\}, \qquad b \in A,$$

where, for elements a, b and c in A, the Jordan triple product is defined by

$$\{abc\} = a \circ (b \circ c) - b \circ (c \circ a) + c \circ (a \circ b).$$

With $\leq$ we denote the linear order relation in A induced by the cone A_+. The zero, resp. the unit, in the algebra A is the least, resp. the greatest, element in the poset $(U(A), \leq)$. To each element a in A there is a smallest idempotent $r(a)$ with

$$r(a) \circ a = a;$$

$r(a)$ is said to be the *support* of a. For convenience we denote with a' the element $e - a$. Then the mapping $'$: $U(A) \to U(A)$ is an orthocomplementation which makes $(U(A), \leq, ')$ into a complete orthomodular lattice. We have, for elements p and q in $U(A)$,

$$\begin{aligned} p \leq q &\Leftrightarrow U_p U_q = U_p, \\ p \perp q &\Leftrightarrow U_p U_q = 0, \\ p\, C\, q &\Leftrightarrow U_p U_q = U_q U_p. \end{aligned}$$

Moreover, if $p\, C\, q$ then

$$p \wedge q = U_p q = p \circ q$$

and if $p \perp q$ then

$$p \vee q = p + q$$

Let D be a non-empty subset of $U(A)$ then

$$\bigvee D = \text{weak*-lim}\left(r\left(\sum_{p \in C} p\right)\right)_{C \in D^f, C \neq \emptyset}.$$

For details the reader is referred to [1, 8, 20].

In what follows we consider A_* to be embedded into the dual Banach space A^* of A under the evaluation map, a linear isometry. An element x in A_* is called a *normal state* on A if $x(a)$ is positive for all positive elements a in A and $x(e)$ equals 1. The convex set $S(A)$ of normal states on A is referred to as the *normal state space* of A. Denote by $\Phi(x)$ the restriction to $U(A)$ of the normal state x. Then clearly $\Phi(x)$ is a probability measure on the orthomodular poset $(U(A), \leq, ')$. Moreover, a normal state x being a weak*-continuous linear functional on A, it follows that $\Phi(x)$ is completely additive. Unitalty of the convex subset $\Phi(S(A))$ of $\Omega(U(A))$ follows from the fact that every JBW-algebra has a non-empty normal state space.

Generally, let a be a positive element smaller than e. Using spectral theory one shows that the sequence $(a_n)_{n\in\mathbb{N}}$ defined by

$$a_n := ((a')^n)',$$

is isotone with upper bound e and converges in the weak*-topology to the support $r(a)$ of a.

Now, let x be a normal state which vanishes on the idempotents p and q. In the sequence above, equate a with $1/2p + 1/2q$. By the Cauchy-Schwarz inequality it follows that $x(a_{2n})$ equals zero for all natural numbers n. Therefore

$$0 = x(r(1/2p + 1/2q)) = x(r(p + q)) = x(p \vee q).$$

Together with Theorem 2.1.(ii) we conclude that $\Phi(S(A))$ is a subset of $\Omega_{c\text{-}JP}(U(A))$.

Let x be a normal state and p an idempotent such that $x(p)$ is different from 0. Then the functional defined by

$$x_p(a) := x(U_p a)/x(p), \qquad a \in A,$$

is a normal state on A since U_p is a weak*-continuous positive map on A. It follows at once that $\Phi(S(A))$ is closed under central conditioning.

Anticipating the fact that a JBW-algebra is associative if and only if the mappings U_p, $p \in U(A)$, commute pairwise we get the following corollary to the Theorems 3.4 and 3.5:

COROLLARY 5.1. *A* JBW-*algebra is associative if and only if the collection of normal states, considered as probability measures on the orthomodular poset of idempotents, has the positive Lebesgue decomposition property.*

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